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Scaling limits of the fuzzy sphere at one loop

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Abstract: We study the one loop dynamics of QFT on the fuzzy sphere and calculate the planar and nonplanar contributions to the two point function at one loop. We show that there is no UV/IR mixing on the fuzzy sphere. The fuzzy sphere is characterized by two moduli: a dimensionless parameter N and a dimensionful radius R. Different geometrical phases can obtained at different corners of the moduli space. In the limit of the commutative sphere, we find that the two point function is regular without UV/IR mixing; however quantization does not commute with the commutative limit, and a finite "noncommutative anomaly" survives in the commutative limit. In a different limit, the noncommutative plane $\mathbb{R}^2_\theta$ is obtained, and the UV/IR mixing reappears. This provides an explanation of the UV/IR mixing as an infinite variant of the "noncommutative anomaly".

Keywords: Field Theories in Lower Dimensions, M-Theory, D-branes.

Contents

1. Introduction

Much effort has been spent in recent years to study quantum field theory on noncommutative spaces. There are many reasons why this is of interest, most of which are related to our poor understanding of physics and the nature of spacetime at very short distances. Additional motivation came from the possibility to realize such spaces in string theory or M-theory [1]–[4]. This provides new insights to issues such as nonlocality and causality of a noncommutative field theory, which are crucial in understanding the structure of quantum spacetime.

Some of the problems that can arise in QFT on noncommutative spaces are illustrated in the much-studied example of the noncommutative plane $\mathbb{R}^n_\theta$; see [5] for a recent review. One of the most intriguing phenomena on that space is the existence of an ultraviolet/infrared (UV/IR) mixing [6] in the quantum effective action. Due to this mixing, an IR singularity arises from integrating out the UV degrees of freedom. This threatens the renormalizability and even the existence of a QFT. Hence a better understanding (beyond the technical level) of the mechanism of UV/IR mixing and possible ways to resolve it are certainly highly desirable. One possible approach is to approximate $\mathbb{R}^n_\theta$ in terms of a different noncommutative space. We realize this idea in the present paper, approximating $\mathbb{R}^2_\theta$ by a fuzzy sphere. This will allow to understand the UV/IR mixing as an infinite variant of a "noncommutative anomaly" on the fuzzy sphere, which is a closely related but different phenomenon discussed below. This is one of our main results. A related, but less geometric approach was considered in [7].

In this article, we consider scalar Φ^4 theory on the fuzzy sphere, and calculate the two point function at one loop. The fuzzy sphere S^2_N is a particularly simple noncommutative space [8], characterized by its radius R and a "noncommutativity" parameter N which is an integer. It approaches the classical sphere in the limit $N \to \infty$ for fixed R, and can be thought of as consisting of N "quantum cells". The algebra of functions on S^2_N is finite, with maximal angular momentum N. Nevertheless, it admits the full symmetry group SO(3) of motions. The fuzzy sphere is closely related to several other noncommutative spaces [9]. In particular, it can be used as an approximation to the quantum plane $\mathbb{R}^2_\theta$, by "blowing up" for example the neighborhood of the south pole. Thus QFT on S^2_N should provide an approximation of the QFT on the quantum plane.

The fuzzy sphere has the additional merit that it is very clear how to quantize field theory on it, using a finite analog of the path integral [10]. Therefore QFT on this space is a priori completely well–defined, on a mathematical level. Nevertheless, it is not clear at all whether such a theory makes sense from a physical point of view, i.e. whether there exists a limiting theory for large N, which could be interpreted as a QFT on the classical sphere. There might be a similar UV/IR problem as on the quantum plane $\mathbb{R}^2_\theta$, as was claimed in a recent paper [11]. In other words, it is not clear if and in what sense such a QFT is renormalizable. As a first step, we calculate in the present paper the two point function at one loop and find that it is well defined and regular, without UV/IR mixing. Moreover, we find a closed formula for the two point function in the commutative limit, i.e. we calculate the leading term in a $1/N$ expansion.

It turns out that the 1-loop effective action on S^2_N in the commutative limit differs from the 1-loop effective action on the commutative sphere S^2 by a finite term, which we call "noncommutative anomaly" (NCA). It is a mildly nonlocal, "small" correction to the kinetic energy on S^2, and changes the dispersion relation. It arises from the nonplanar loop integration. Finally, we consider the planar limit of the fuzzy sphere. We find that a IR singularity is developed in the nonplanar two point function, and hence the UV/IR mixing emerges in this limit. This provides an understanding of the UV/IR mixing for QFT on $\mathbb{R}^2_\theta$ as a "noncommutative anomaly" which becomes singular in the planar limit of the fuzzy sphere dynamics.

This paper is organized as follows. In section 2, we consider different geometrical limits of the classical (ie. $\hbar = 0$) fuzzy sphere. In particular, we show how the commutative sphere and the noncommutative plane $\mathbb{R}^2_\theta$ can be obtained in different corners of the moduli space of the fuzzy sphere. In section 3, we study the quantum effects of scalar Φ^4 field theory on the fuzzy sphere at 1-loop. We show that the planar and nonplanar 2-point function are both regular in the external angular momentum and no IR singularity is developed. This means that there is no UV/IR mixing phenomenon on the fuzzy sphere. We also find that the planar and nonplanar two point functions differ by a finite amount which is smooth in the external angular momentum, and survives in the commutative limit. Therefore the commutative limit of the Φ^4 theory at one loop differs from the corresponding one loop quantum theory on the commutative sphere by a finite term (3.23). In section 4, we consider the planar limit of this QFT, and recover the UV/IR mixing.

2. The fuzzy sphere and some limits

2.1 The fuzzy sphere S_N^2

We start by recalling the definition of fuzzy sphere in order to fix our conventions and notation. The algebra of functions on the fuzzy sphere is the finite algebra S_N^2 generated by hermitian operators $x = (x_1, x_2, x_3)$ satisfying the defining relations

$$[x_i, x_j] = i\lambda_N \epsilon_{ijk} x_k \,, \tag{2.1}$$

$$x_1^2 + x_2^2 + x_3^2 = R^2 \,. \tag{2.2}$$

The noncommutativity parameter λ_N is of dimension length, and can be taken positive. The radius R is quantized in units of λ_N by

$$\frac{R}{\lambda_N} = \sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}\,, \quad N = 1, 2, \ldots \tag{2.3}$$

This quantization can be easily understood. Indeed (2.1) is simply the Lie algebra $su(2)$, whose irreducible representation are labeled by the spin $\alpha := N/2$. The Casimir of the spin-$N/2$ representation is quantized, and related to R^2 by (2.2). Thus the fuzzy sphere is characterized by its radius R and the "noncommutativity parameters" N or λ_N. The algebra of "functions" S_N^2 is simply the algebra $Mat(N+1)$ of $(N+1)\times(N+1)$ matrices. It is covariant under the adjoint action of SU(2), under which it decomposes into the irreducible representations with dimensions $(1)\oplus(3)\oplus(5)\oplus\ldots\oplus(2N+1)$.

The integral of a function $F \in S_N^2$ over the fuzzy sphere is given by

$$R^2 \int F = \frac{4\pi R^2}{N+1}\mathrm{tr}[F(x)]\,, \tag{2.4}$$

where we have introduced $\int$, the integral over the fuzzy sphere with unit radius. It agrees with the integral $\int d\Omega$ on S^2 in the large N limit. Invariance of the integral under the rotations SU(2) amounts to invariance of the trace under adjoint action. One can also introduce the inner product

$$(F_1, F_2) = \int F_1 \dagger F_2 \,. \tag{2.5}$$

A complete basis of functions on S_N^2 is given by the $(N+1)^2$ spherical harmonics, $Y_j^J, (J = 0, 1, \ldots, N; -J \leq j \leq J)$,[1] which are the weight basis of the spin J component of S_N^2 explained above. They correspond to the usual spherical harmonics, however the angular momentum has an upper bound N here. This is a characteristic feature of fuzzy sphere. The normalization and reality for these matrices can be taken to be

$$(Y_j^J, Y_{j'}^{J'}) = \delta_{JJ'}\delta_{jj'}, \qquad (Y_j^J)^\dagger = (-1)^J Y_{-j}^J \,. \tag{2.6}$$

They obey the "fusion" algebra

$$\begin{aligned} Y_i^I Y_j^J = & \sqrt{\frac{N+1}{4\pi}} \sum_{K,k} (-1)^{2\alpha+I+J+K+k} \sqrt{(2I+1)(2J+1)(2K+1)} \begin{pmatrix} I & J & K \\ i & j & -k \end{pmatrix} \times \\ & \times \begin{Bmatrix} I & J & K \\ \alpha & \alpha & \alpha \end{Bmatrix} Y_k^K \,, \end{aligned} \tag{2.7}$$

where the sum is over $0 \leq K \leq N, -K \leq k \leq K$, and

$$\alpha = N/2 \,. \tag{2.8}$$

Here the first bracket is the Wigner $3j$-symbol and the curly bracket is the $6j$-symbol of $su(2)$, in the standard mathematical normalization [12]. Using the Biedenharn–Elliott identity (B.4), it is easy to show that (2.7) is associative. In particular, $Y_0^0 = \frac{1}{\sqrt{4\pi}} \mathbf{1}$. The relation (2.7) is independent of the radius R, but depends on the deformation parameter N. It is a deformation of the algebra of product of the spherical harmonics on the usual sphere. We will need (2.7) to derive the form of the propagator and vertices in the angular momentum basis.

Now we turn to various limits of the fuzzy sphere. By tuning the parameters R and N, one can obtain different limiting algebras of functions. In particular, we consider the commutative sphere S^2 and the noncommutative plane $\mathbb{R}_\theta^2$.

2.2 The commutative sphere limit S^2

The commutative limit is defined by

$$N \to \infty, \quad \text{keeping } R \text{ fixed} \,. \tag{2.9}$$

In this limit, (2.1) reduces to $[x_i, x_j] = 0$ and we obtain the commutative algebra of functions on the usual sphere S^2. Note that (2.7) reduces to the standard product of

[1] We will use capital and small letter (e.g. (J, j)) to refer to the eigenvalue of the angular momentum operator $\mathbf{J^2}$ and J_z respectively.

spherical harmonics, due to the asymptotic relation between the $6j$-symbol and the Wigner $3j$-symbol [12],

$$\lim_{\alpha\to\infty}(-1)^{2\alpha}\sqrt{2\alpha}\begin{Bmatrix} I & J & K \\ \alpha & \alpha & \alpha \end{Bmatrix} = (-1)^{I+J+K}\begin{pmatrix} I & J & K \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.10}$$

2.3 The quantum plane limit $\mathbf{R}^2_\theta$

If the fuzzy sphere is blown up around a given point, it becomes an approximation of the quantum plane [8]. To obtain this planar limit, it is convenient to first introduce an alternative representation of the fuzzy sphere in terms of stereographic projection. Consider the generators

$$y_+ = 2Rx_+(R-x_3)^{-1}, \qquad y_- = 2R(R-x_3)^{-1}x_-, \tag{2.11}$$

where $x_\pm = x_1 \pm ix_2$. The generators $y_\pm$ are the coordinates of the stereographic projection from the north pole. $y=0$ corresponds to the south pole. Now we take the large N and large R limit, such that

$$N\to\infty, \quad R^2 = N\theta/2 \to\infty, \quad \text{keeping } \theta \text{ fixed}. \tag{2.12}$$

In this limit,

$$\frac{\lambda_N}{\sqrt{\theta}} \sim \frac{1}{\sqrt{N}} \tag{2.13}$$

and $[y_+,y_-] = -4R^2\lambda_N(R-x_3)^{-1} + o(\lambda_N^2)$. Since $y_+y_- = 4R^2(R+x_3)(R-x_3)^{-1} + o(N^{-1/2})$, we can cover the whole y-plane with $x_3 = -R+\beta/R$ with finite but arbitrary β. The commutation relation of the y generators takes the form

$$[y_+,y_-] = -2\theta \tag{2.14}$$

up to corrections of order λ_N^2, or

$$[y_1,y_2] = -i\theta \tag{2.15}$$

with $y_\pm = y_1 \pm iy_2$.

3. One loop dynamics of Φ^4 on the fuzzy sphere

Consider a scalar Φ^4 theory on the fuzzy sphere, with action

$$S_0 = \int \frac{1}{2}\,\Phi(\Delta+\mu^2)\Phi + \frac{g}{4!}\Phi^4. \tag{3.1}$$

Here Φ is hermitian, μ^2 is the dimensionless mass square, g is a dimensionless coupling and $\Delta = \sum J_i^2$ is the Laplace operator. The differential operator J_i acts on function $F\in S_N^2$ as

$$J_iF = \frac{1}{\lambda_N}[x_i,F]. \tag{3.2}$$

This action is valid for any radius R, since μ and g are dimensionless. To quantize the theory, we will follow the path integral quantization procedure as explained in [10]. We expand Φ in terms of the modes,

$$\Phi = \sum_{L,l} a_l^L Y_l^L, \quad a_l^L{}^\dagger = (-1)^l a_{-l}^L \,. \tag{3.3}$$

The Fourier coefficient a_l^L are then treated as the dynamical variables, and the path integral quantization is defined by integrating over all possible configuration of a_l^L. Correlation functions are computed using [10]

$$\langle a_{l_1}^{L_1} \cdots a_{l_k}^{L_k} \rangle = \frac{\int [\mathcal{D}\Phi] e^{-S_0} \; a_{l_1}^{L_1} \cdots a_{l_k}^{L_k}}{\int [\mathcal{D}\Phi] e^{-S_0}} \,. \tag{3.4}$$

For example, the propagator is

$$\langle a_l^L a_{l'}^{L'}{}^\dagger \rangle = (-1)^l \langle a_l^L a_{-l'}^{L'} \rangle = \delta_{LL'} \delta_{ll'} \frac{1}{L(L+1)+\mu^2} \,, \tag{3.5}$$

and the vertices for the Φ^4 theory are given by

$$a_{l_1}^{L_1} \cdots a_{l_4}^{L_4} \; V(L_1, l_1; \cdots; L_4, l_4) \,, \tag{3.6}$$

where

$$\begin{aligned} V(L_1, l_1; \cdots; L_4, l_4) = \frac{g}{4!} \frac{N+1}{4\pi} (-1)^{L_1+L_2+L_3+L_4} \prod_{i=1}^{4} (2L_i+1)^{1/2} \sum_{L,l} (-1)^l (2L+1) \cdot \\ \cdot \begin{pmatrix} L_1 & L_2 & L \\ l_1 & l_2 & l \end{pmatrix} \begin{pmatrix} L_3 & L_4 & L \\ l_3 & l_4 & -l \end{pmatrix} \begin{Bmatrix} L_1 & L_2 & L \\ \alpha & \alpha & \alpha \end{Bmatrix} \begin{Bmatrix} L_3 & L_4 & L \\ \alpha & \alpha & \alpha \end{Bmatrix} . \end{aligned} \tag{3.7}$$

One can show that V is symmetric with respect to cyclic permutation of its arguments (L_i, l_i).

The $1PI$ two point function at one loop is obtained by contracting 2 legs in (3.7) using the propagator (3.5). The planar contribution is defined by contracting neighboring legs:

$$\begin{aligned} (\Gamma^{(2)}_{\text{planar}})^{LL'}_{ll'} &= \frac{g}{4\pi} \frac{1}{3} \, \delta_{LL'} \delta_{l,-l'} (-1)^l \cdot I^P \,, \\ I^P &:= \sum_{J=0}^{N} \frac{2J+1}{J(J+1)+\mu^2} \,. \end{aligned} \tag{3.8}$$

All 8 contributions are identical. Similarly by contracting non-neighboring legs, we find the non-planar contribution

$$\begin{aligned} (\Gamma^{(2)}_{\text{nonplanar}})^{LL'}_{ll'} &= \frac{g}{4\pi} \frac{1}{6} \, \delta_{LL'} \delta_{l,-l'} (-1)^l \cdot I^{NP} \,, \\ I^{NP} &:= \sum_{J=0}^{N} (-1)^{L+J+2\alpha} \, \frac{(2J+1)(2\alpha+1)}{J(J+1)+\mu^2} \begin{Bmatrix} \alpha & \alpha & L \\ \alpha & \alpha & J \end{Bmatrix} . \end{aligned} \tag{3.9}$$

Again the 4 possible contractions agree. These results can be found using standard identities for the $3j$ and $6j$ symbols, see e.g. [12] and appendix B.

It is instructive to note that I^{NP} can be written in the form

$$I^{NP} = \sum_{J=0}^{N} \frac{2J+1}{J(J+1)+\mu^2}\, f_J\,, \tag{3.10}$$

where f_J is obtained from the generating function

$$\begin{aligned} f(x) &= \sum_{J=0}^{\infty} f_J x^J \\ &= \frac{1}{1-x}\, {}_2F_1\left(-L, L+1, 2\alpha+2, \frac{x}{x-1}\right) {}_2F_1\left(-L, L+1, -2\alpha, \frac{x}{x-1}\right)\,. \end{aligned}$$

Here the hypergeometric function ${}_2F_1(-L, L+1; c; z)$ is a polynomial of degree L for any c. Note that the oscillatory sign in I^{NP} in (3.9) is cancelled by the sign of the $6j$-symbol in (3.16), and is replaced by a slower oscillatory behaviour of the $6j$-symbol as a function of L and J. The latter is precisely the counter-part of the nonplanar Moyal phase factor in the noncommutative plane case.[2]

For example, for $L = 0$, one obtains

$$f_J = 1, \quad 0 \le J \le N\,, \tag{3.11}$$

and hence the planar and nonplanar two point functions coincides. For $L = 1$, we have

$$f_J = 1 - \frac{J(J+1)}{2\alpha(\alpha+1)}, \quad 0 \le J \le N \;, \tag{3.12}$$

and hence

$$I^{NP} = I^P - \frac{1}{2\alpha(\alpha+1)} \sum_{J=0}^{2\alpha} \frac{J(J+1)(2J+1)}{J(J+1)+\mu^2}\,. \tag{3.13}$$

Note that the difference between the planar and nonplanar two point functions is finite. It is easy to convince oneself that for any finite external angular momentum L, the difference between the planar and nonplanar two point function is finite and analytic in $1/\alpha$. This fact is important as it implies that, unlike in the $\mathbb{R}^n_\theta$ case, there is no infrared singularity developed in the nonplanar amplitude. We will have more to say about this later.

[2]It was argued in [11] that the nonplanar two-point function has a different sign for even and odd external angular momentum L. This is not correct, because as we just explained, the oscillations are indeed much milder after combining with the $6j$-symbol. As we will show, this leads to a well-behaved loop integral which is a sum over all J, and the resulting low L-behaviour is completely regular. We note that only the single value of $J = 2\alpha$ was considered in [11].

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3.1 On UV/IR mixing and the commutative limit

Let us recall that in the case of noncommutative space $\mathbb{R}^n_\theta$, the one-loop contribution to the effective action often develops a singularity at $\theta p = 0$ [6, 14]. This infrared singularity is generated by integrating out the infinite number of degree of freedom in the nonplanar loop. This phenomenon is referred to as "UV/IR mixing", and it implies in particular that (1) the nonplanar amplitude is singular when the external momentum is zero in the noncommutative directions; and (2) the quantum effective action in the commutative limit is different from the quantum effective action of the commutative limit [13].

Effective action on the fuzzy sphere. We want to understand the behavior of the corresponding planar and nonplanar two point functions on the fuzzy sphere, to see if there is a similar UV/IR phenomenon. We emphasize that this is not obvious a priori even though quantum field theory on the fuzzy sphere is always finite. The question is whether the 2-point function is smooth at small values of L, or rapidly oscillating as was indeed claimed in a recent paper [11]. Integrating out all the degrees of freedom in the loop could in principle generate a IR singularity, for large N.

However, this is not the case. We found above that the planar and nonplanar two point function agree precisely with each other when the external angular momentum $L = 0$. For general L, a closed expression for f_J for general L is difficult to obtain. We will derive below an approximate formula for the difference $I^{NP} - I^P$, which is found to be an excellent approximation for large N by numerical tests, and becomes exact in the commutative limit $N \to \infty$.

First, the planar contribution to the two point function

$$I^P = \sum_{J=0}^{N} \frac{2J+1}{J(J+1)+\mu^2} \tag{3.14}$$

agrees precisely with the corresponding terms on the classical sphere as $N \to \infty$, and it diverges logarithmically

$$I^P \sim \log\alpha + o(1)\,. \tag{3.15}$$

To understand the nonplanar contribution, we start with the following approximation formula [12] for the $6j$ symbols due to Racah,

$$\begin{Bmatrix} \alpha & \alpha & L \\ \alpha & \alpha & J \end{Bmatrix} \approx \frac{(-1)^{L+2\alpha+J}}{2\alpha} P_L\Big(1 - \frac{J^2}{2\alpha^2}\Big)\,, \tag{3.16}$$

where P_L are the Legendre Polynomials. This turns out to be an excellent approximation for all $0 \le J \le 2\alpha$, provided α is large and $L \ll \alpha$. Since this range of validity of this approximation formula is crucial for us, we shall derive it in appendix A. This allows then to rewrite the sum in (3.9) to a very good approximation as

$$I^{NP} - I^{P} = \sum_{J=0}^{2\alpha} \frac{2J+1}{J(J+1)+\mu^2} \left(P_L(1 - \frac{J^2}{2\alpha^2}) - 1 \right) \tag{3.17}$$

for large α. Since $P_L(1) = 1$ for all L, only $J \gg 1$ contributes, and one can approximate the sum by the integral

$$\begin{aligned} I^{NP} - I^{P} &\approx \int_0^2 du \, \frac{2u + 1/\alpha}{u^2 + u/\alpha + \mu^2/\alpha^2} \left(P_L \left(1 - \frac{u^2}{2} \right) - 1 \right) \\ &= \int_{-1}^{1} dt \, \frac{1}{1-t} (P_L(t) - 1) + o(1/\alpha) \, , \end{aligned} \tag{3.18}$$

assuming $\mu \ll \alpha$. This integral is finite for all L. Indeed using generating functions techniques, it is easy to show that

$$\int_{-1}^{1} dt \frac{1}{1-t} (P_L(t) - 1) = -2 \left(\sum_{k=1}^{L} \frac{1}{k} \right) = -2h(L) \, , \tag{3.19}$$

where $h(L) = \sum_{k=1}^{L} \frac{1}{k}$ is the harmonic number and $h(0) = 0$. While $h(L) \approx logL$ for large L, it is finite and well-behaved for small L. Therefore we obtain the effective action

$$S_{one-loop} = S_0 + \int \frac{1}{2} \, \Phi(\delta\mu^2 - \frac{g}{12\pi} h(\widetilde{\Delta}))\Phi + o(1/\alpha) \tag{3.20}$$

to the first order in the coupling where

$$\delta\mu^2 = \frac{g}{8\pi} \sum_{J=0}^{N} \frac{2J+1}{J(J+1)+\mu^2} \tag{3.21}$$

is the mass square renormalization, and $\widetilde{\Delta}$ is the function of the laplacian which has eigenvalues L on Y_l^L. Thus we find that the effects due to noncommutativity are analytic in the noncommutative parameter $1/\alpha$. This is a finite quantum effect with nontrivial, but mild L dependence. Therefore no IR singularity is developed, and we conclude that there is no UV/IR problem on the fuzzy sphere.[3]

[3]The author of [11] adopted a wilsonian approach integrating the "cutoff" by one unit, and argued that the effective action is not a smooth function of the external momentum and suggested this to be a signature of UV/IR mixing. We disagree with his result. In this paper, we follow the more conventional program of renormalization (for the 2-point function), and calculate the full loop integral which is perfectly regular.

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The commutative limit. The commutative limit of the QFT is defined by the limit

$$\alpha \to \infty, \quad \text{keeping } R,\ g,\ \mu \text{ fixed}\,. \tag{3.22}$$

In this limit, the resulting one-loop effective action differs from the effective action obtained by quantization on the commutative sphere by an amount

$$\Gamma^{(2)}_{NCA} = -\frac{g}{24\pi}\int \Phi h(\widetilde{\Delta})\Phi\,. \tag{3.23}$$

We refer to this as a "NonCommutative Anomaly", since it is the piece of the quantum effective action which is slightly nonlocal and therefore not present in the classical action. "Noncommutative" also refers to fact that the quantum effective action depends on whether we quantize first or take the commutative limit first.

The new term $\Gamma^{(2)}_{NCA}$ modifies the dispersion relation on the fuzzy sphere. It is very remarkable that such a "signature" of an underlying noncommutative space exists, even as the noncommutativity on the geometrical level is sent to zero. A similar phenomena is the induced Chern-Simon term in 3-dimensional gauge theory on $\mathbb{R}^3_\theta$ [13]. This has important implications on the detectability of an underlying noncommutative structure. The reason is that the vacuum fluctuations "probe" the structure of the space even in the UV, and depend nontrivially on the external momentum in the nonplanar diagrams. Higher-order corrections may modify the result. However since the theory is completely well-defined for finite N, the above result (3.23) is meaningful for small coupling g.

Summarizing, we find that quantization and taking the commutative limit does not commute on the fuzzy sphere, a fact which we refer to as "noncommutative anomaly". A similar phenomenon also occurs on the noncommutative quantum plane $\mathbb{R}^n_\theta$. However, in contrast to the case of the quantum plane, the "noncommutative anomaly" here is not due to UV/IR mixing since there is no UV/IR mixing on the fuzzy sphere. We therefore suggest that the existence of a "noncommutative anomaly" is a generic phenomenon and is independent of UV/IR mixing.[4] One can expect that the "noncommutative anomaly" does not occur for supersymmetric theories on the 2–sphere.

4. Planar limit of quantum Φ^4

In this section, we study the planar limit of the Φ^4 theory on the fuzzy sphere at one loop. Since we have shown that there is no UV/IR mixing on the fuzzy sphere, one may wonder whether (4.9) could provide a regularization for the nonplanar two point

[4]However, as we will see, they are closely related. We will show in the next section that the UV/IR mixing on the noncommutative plane arises in the planar limit of the "NCA" for the fuzzy sphere.

function (4.10) on $\mathbb{R}^2_\theta$ which does not display an infrared singularity. This would be very nice, as this would mean that UV/IR can be understood as an artifact that arises out of a bad choice of variables. However, this is not the case.

To take the planar limit, we need in addition to (2.12), also

$$\mu^2 = m^2 R^2 \sim \alpha \to \infty, \quad \text{keeping } m \text{ fixed}, \tag{4.1}$$

so that a massive scalar theory is obtained. We wish to identify in the limit of large R the modes on the sphere with angular momentum L with modes on the plane with linear momentum p. This can be achieved by matching the laplacian on the plane with that on the sphere in the large radius limit, ie.

$$L(L+1)/R^2 = p^2 . \tag{4.2}$$

It follows that

$$p = \frac{L}{R} . \tag{4.3}$$

Note that by (2.12), a mode with a fixed nonzero p corresponds to a mode on the sphere with large L:

$$L \sim R \sim \sqrt{\alpha}. \tag{4.4}$$

Since L is bounded by α, there is a UV cutoff Λ on the plane at

$$\Lambda = \frac{2\alpha}{R} . \tag{4.5}$$

Denote the external momentum of the two point function by p. It then follows that $\alpha \gg L \gg 1$ as long as $p \neq 0$.

It is easy to see that the planar amplitude (3.8) becomes

$$I^P = 2 \int_0^\Lambda dk \frac{k}{k^2 + m^2} \tag{4.6}$$

in the quantum plane limit, with $k = J/R$. This is precisely the planar contribution to the two point function on $\mathbb{R}^2_\theta$.

For the nonplanar two point function (3.9), we can again use the formula (3.16) which is valid for all J and large α, since the condition $\alpha \gg L$ is guaranteed by (4.4). Therefore

$$I^{NP}(p) = 2 \int_0^\Lambda dk \; \frac{k}{k^2 + m^2} P_{pR} \left(1 - 2\frac{k^2}{\Lambda^2} \right) \tag{4.7}$$

For large $L = pR$, we can use the approximation formula

$$P_L(\cos\phi) = \sqrt{\frac{\phi}{\sin\phi}} J_0((L+1/2)\phi) \quad + \; O(L^{-3/2}), \tag{4.8}$$

which is uniformly convergent [15] as $L \to \infty$ in the interval $0 \le \phi \le \pi - \epsilon$ for any small, but finite $\epsilon > 0$. Then one obtains

$$\begin{aligned} I^{NP}(p) &\approx 2 \int_0^\Lambda dk \, \frac{k}{k^2+m^2} \sqrt{\frac{\phi_k}{\sin\phi_k}} \, J_0(pR\phi_k) \\ &\approx 2 \int_0^\Lambda dk \, \frac{k}{k^2+m^2} \, J_0(\theta p k), \end{aligned} \tag{4.9}$$

where $\phi_k = 2\arcsin(k/\Lambda)$. The singularity at $\phi = \pi$ on the rhs of (4.8) (which is an artefact of the approximation and not present in the lhs) is integrable and does not contribute to (4.9) for large $p\Lambda\theta$. The integrals in (4.9) are (conditionally) convergent for $p \neq 0$, and the approximations become exact for $p\Lambda\theta \to \infty$. Therefore we recover precisely the same form as the one loop nonplanar two point function on $\mathbb{R}^2_\theta$,

$$\frac{1}{2\pi} \int_0^\Lambda d^2k \, \frac{1}{k^2+m^2} e^{i\theta p \times k} \,. \tag{4.10}$$

For small $p\Lambda\theta$, i.e. in the vicinity of the induced infrared divergence on $\mathbb{R}^2_\theta$, these approximations are less reliable. We can obtain the exact form of the infrared divergence from (3.19),

$$I^{NP} = -2\log(p\sqrt{\theta}) + (I^P - \log\alpha) \,. \tag{4.11}$$

Hence we find the same logarithmic singularity in the infrared as on $\mathbb{R}^2_\theta$ [6]. In other words, we find that the UV/IR mixing phenomenon which occurs in QFT on $\mathbb{R}^2_\theta$ can be understood as the infinite limit of the noncomutative anomaly (3.23) on the fuzzy sphere. Hence one could use the fuzzy sphere as a regularization of $\mathbb{R}^2_\theta$, where the logarithmic singularity $\log(p\sqrt{\theta})$ gets "regularized" by (3.19).

5. Discussion

We have done a careful analysis of the one-loop dynamics of scalar Φ^4 theory on the fuzzy sphere S^2_N. We found that the two point function is completely regular, without any UV/IR mixing. We also give a closed expression for the two point function in the commutative limit, i.e. we find an exact form for the leading term in a $1/N$ expansion. Using this we discover a "noncommutative anomaly" (NCA), which characterizes the difference between the quantum effective action on the commutative sphere S^2 and the commutative limit $N \to \infty$ of the quantum effective action on the fuzzy sphere. This anomaly is finite but mildly nonlocal on S^2, and changes the dispersion relation. It arises from the nonplanar loop integration.

It is certainly intriguing and perhaps disturbing that even an "infinitesimal" quantum structure of (space)time has a finite, nonvanishing effect on the quantum theory. Of course this was already found in the UV/IR phenomenon on $\mathbb{R}^n_\theta$, however in that case one might question whether the quantization procedure based on deformation quantization is appropriate. On the fuzzy sphere, the result is completely well-defined and unambiguous. One might argue that a "reasonable" QFT should be free of such a NCA, so that the effective, macroscopic theory is insensitive to small variations of the structure of spacetime at short distances. On the other hand, it is conceivable that our world is actually noncommutative, and the noncommutative dynamics should be taken seriously. Then there is no reason to exclude theories with NCA. In particular, one would like to understand better how sensitive these "noncommutative anolamies" are to the detailed quantum structure of spacetime.

By approximating the QFT on $\mathbb{R}^2_\theta$ with the QFT on the fuzzy sphere, we can explain the UV/IR mixing from the point of view of the fuzzy sphere as a infinite variant of the NCA. In some sense, we have regularized $\mathbb{R}^2_\theta$. It would be interesting to provide an explanation of the UV/IR mixing also for the higher dimensional case $\mathbb{R}^4_\theta$. To do this, the first step is to realize $\mathbb{R}^4_\theta$ as a limit of a "nicer" noncommutative manifold. A first candidate is the product of two fuzzy spheres. Much work remains to be done to clarify this situation.

It would also be very desirable to include fermions and gauge fields in these considerations. In particular it will be interesting to determine the dispersion relation for "photons", depending on the "fuzzyness" of the underlying geometry. In the case of noncommutative QED on $\mathbb{R}^4_\theta$, this question was studied in [14], where a nontrivial modification to the dispersion relation of the "photon" was found which makes the theory ill–defined. In view of our results, one may hope that these modifications are milder on the fuzzy sphere and remain physically sensible.

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A. Approximation formula (3.16)

We derive the approximation formula (3.16) for large α and $0 \leq J \leq 2\alpha$, assuming $L \ll \alpha$.

There is an exact formula for the $6j$ coefficients due to Racah (see e.g. [12]), which can be written in the form

$$\begin{Bmatrix} \alpha & \alpha & L \\ \alpha & \alpha & J \end{Bmatrix} = (-1)^{2\alpha+J} \sum_n (-1)^n \binom{L}{n}^2 \times \\ \times \frac{(2\alpha-L)!(2\alpha+J+n+1)!(2\alpha-J)!(J!)^2}{(2\alpha+L+1)!(2\alpha+J+1)!(2\alpha-J-n)!((J-L+n)!)^2}\,.$$

The sum is from $n = \max\{0, L-J\}$ to $\min\{L, 2\alpha - J\}$, so that all factorials are non-negative. Assume first that $L \leq J \leq 2\alpha - L$, so that the sum is from 0 to L. Since $\alpha \gg L$, this becomes

$$\begin{Bmatrix} \alpha & \alpha & L \\ \alpha & \alpha & J \end{Bmatrix} \approx (-1)^{2\alpha+J} \frac{1}{(2\alpha)^{2L+1}} \sum_{n=0}^{L} (-1)^n \binom{L}{n}^2 (4\alpha^2 - J^2)^n \left(\frac{J!}{(J-(L-n))!} \right)^2, \tag{A.1}$$

dropping corrections of order $o(\frac{L}{\alpha})$. Now there are 2 cases: either $J \gg L$, or otherwise $J \ll \alpha$ since $\alpha \gg L$. Consider first

1. $J \gg L$:

   Then $\frac{J!}{(J-(L-n))!}$ can be replaced by J^{L-n}, up to corrections of order $o(\frac{L}{J})$. Therefore

$$\begin{aligned} \begin{Bmatrix} \alpha & \alpha & L \\ \alpha & \alpha & J \end{Bmatrix} &\approx \frac{(-1)^{2\alpha+J}}{2\alpha} \left(\frac{J}{2\alpha} \right)^{2L} \sum_{n=0}^{L} (-1)^n \binom{L}{n}^2 \left(\left(\frac{2\alpha}{J} \right)^2 - 1 \right)^n \\ &= \frac{(-1)^{2\alpha+J}}{2\alpha} P_L \left(1 - \frac{J^2}{2\alpha^2} \right), \end{aligned} \tag{A.2}$$

   as claimed.

2. $J \ll \alpha$:

   Then in the sum (A.1), the dominant term is $n = L$, because $\frac{J!}{(J-(L-n))!} \leq J^{L-n}$. Therefore one can safely replace the term $\frac{J!}{(J-(L-n))!}$ in this sum by its value at $n = L$, namely J^{L-n}. The remaining terms are smaller by a factor of $(\frac{J}{\alpha})^2$. Hence we can continue as in case 1.

If $J \leq L$, one can either use the same argument as in the 2nd case since the term $n = L$ is dominant, or use the symmetry of the $6j$ symbols in L, J together with $P_L(1 - \frac{J^2}{2\alpha^2}) \approx P_J(1 - \frac{L^2}{2\alpha^2})$ for $J, L \ll \alpha$. Finally if $J + L \geq 2\alpha$, then the term $n = 0$ dominates, and one can proceed as in case 1. Therefore (3.16) is valid for all $0 \leq J \leq 2\alpha$.

One can illustrate the excellent approximation for the $6j$ symbols provided by (3.16) for all $0 \leq J \leq 2\alpha$ using numerical calculations.

B. Some identities of $3j$ and $6j$ symbols

We quote here some identities of the $3j$ and $6j$ symbols which are used to derive the expressions (3.8) and (3.9) for the one-loop corrections. The $3j$ symbols satisfy the orthogonality relation

$$\sum_{j,l} \begin{pmatrix} J & L & K \\ j & l & k \end{pmatrix} \begin{pmatrix} J & L & K' \\ -j & -l & -k' \end{pmatrix} = \frac{(-1)^{K-L-J}}{2K+1} \delta_{K,K'} \delta_{k,k'} , \tag{B.1}$$

assuming that (J, L, K) form a triangle.

The $6j$ symbols satisfy standard symmetry properties, and the orthogonality relation

$$\sum_N (2N+1) \begin{Bmatrix} A & B & N \\ C & D & P \end{Bmatrix} \begin{Bmatrix} A & B & N \\ C & D & Q \end{Bmatrix} = \frac{1}{2P+1}\,\delta_{P,Q}\,, \tag{B.2}$$

assuming that (A, D, P) and (B, C, P) form a triangle. Furthermore, the following sum rule is used in (3.9)

$$\sum_N (-1)^{N+P+Q}(2N+1) \begin{Bmatrix} A & B & N \\ C & D & P \end{Bmatrix} \begin{Bmatrix} A & B & N \\ D & C & Q \end{Bmatrix} = \begin{Bmatrix} A & C & Q \\ B & D & P \end{Bmatrix}. \tag{B.3}$$

The Biedenharn-Elliott relations are needed to verify associativity of (2.7):

$$\sum_N (-1)^{N+S}(2N+1) \begin{Bmatrix} A & B & N \\ C & D & P \end{Bmatrix} \begin{Bmatrix} C & D & N \\ E & F & Q \end{Bmatrix} \begin{Bmatrix} E & F & N \\ B & A & R \end{Bmatrix} = \begin{Bmatrix} P & Q & R \\ E & A & D \end{Bmatrix} \begin{Bmatrix} P & Q & R \\ F & B & C \end{Bmatrix}, \tag{B.4}$$

where $S = A+B+C+D+E+F+P+Q+R$. All these can be found e.g. in [12].

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